★ convolution, and total response

- Last time we saw how to use convolution to find the zero state response, given the impulse response. But why does this work?

<u>Heuristic Derivation of Convolution</u>

- If a system is LTI then

$$a\,x(t-t_0) \rightarrow \boxed{} \rightarrow a\,y(t-t_0)$$

- In particular

$$a\,\delta(t-t_0) \rightarrow \boxed{} \rightarrow a\,h(t-t_0)$$

- AND Let $a = x(t_0)$
- Recall sifting property of δ $\displaystyle\int_{-\infty}^{\infty} f(t)\delta(t-t_0)\,dt = f(t_0)$
- Because $\int$ is a linear operator

$$\int_{-\infty}^{\infty} x(t_0)\,\delta(t-t_0)\,dt \rightarrow \boxed{} \rightarrow \int_{-\infty}^{\infty} x(t_0)\,h(t-t_0)\,dt$$

- Now lets do a change of variables $\tau = t - t_0$
  or $t_0 = t - \tau$
  and $dt = d\tau$

Then $$\int_{-\infty}^{\infty} x(t-\tau)\,\delta(\tau)\,d\tau \rightarrow \boxed{} \rightarrow \int_{-\infty}^{\infty} x(t-\tau)\,h(\tau)\,d\tau$$

- Again, by sifting property

$$x(t) \Rightarrow \boxed{} \Rightarrow \underbrace{\int_{-\infty}^{\infty} x(t-\tau)\,h(\tau)\,d\tau}_{\text{The convolution integral.}} = h * x = x * h$$

★ Summary of time domain analysis.

- To find the total response $y(t)$ to a system with initial conditions and input $x(t)$,

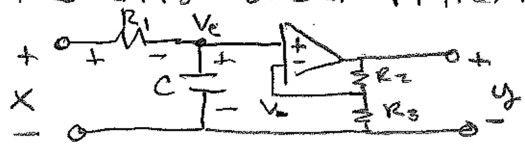
$$y(t) = y_0(t)u(t) + y_{zs}(t) \quad \text{where } Q(D)y = P(D)x$$

- $y_0(t)$ is the solution to $Q(D)y = 0$ subject to I.C.
- $y_{zs}(t) = (h * x)(t)$ where $Q(D)h = \delta(t)$

  or $Q(D)h = 0$ subject to special initial conditions.

- Lets do a simple example all the way through.

Consider the first order filter.



where $V_c(0^-) \neq 0$ and

$x(t) = t e^{-10t} u(t)$

and ideal op-amp.

Find the total Response $y(t)$.

1. Find the Differential Equation description of system.

(1) KCL @ V_c: $\frac{x - V_c}{R_1} = C V_c' \Rightarrow V_c' + \frac{1}{R_1 C} V_c = \frac{1}{R_1 C} x$

(2) KCL @ V_-: $\frac{y - V_-}{R_2} = \frac{V_-}{R_3} \Rightarrow y = \left(1 + \frac{R_2}{R_3}\right) V_-$

(3) OP-Amp: $V_c = V_-$

combining (1)(2)(3) $\Rightarrow$ $y' + \underbrace{\frac{1}{R_1 C}}_{a} y = \underbrace{\frac{\left(1 + \frac{R_2}{R_3}\right)}{R_1 C}}_{b} x$

$y(0^-) = \left(1 + \frac{R_2}{R_3}\right) V_c(0^-) \equiv A$

$y' + ay = bx$

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2. The total response $y(t) = y_0(t) u(t) + y_{zs}(t)$

2.1) Find $y_0(t)$ from $y' + ay = 0$ with $y_0(0^-) = A$

$Q(D) = D + a$ has one root at $-a$, so.

$y_0(t) = C e^{-at} \Rightarrow y_0(0^-) = C = A$

$\therefore \; y_0(t) = A e^{-at}$

2.2) Find $y_{zs}(t)$ from $y' + ay = bx = b t e^{-10t} u(t)$

2.2.1) find $h(t)$ from $y_n' + a y_n = 0 \quad y_n(0^+) = 1$

$y_n(t) = C e^{-at} \Rightarrow y_n(0^+) = C = 1$

$h(t) = \underset{0}{\underset{\shortparallel}{b_0}} \delta(t) + [P(D) y_n] u(t) \qquad P(D) = b$

$= b e^{-at} u(t)$

2.2.2) Find $y_{zs}(t) = (h * x)(t)$

$$h(t) = be^{-at} \qquad x(t) = te^{-10t}u(t)$$

From Table 2.1 No 9.

x_1	x_2	$x_1 * x_2$
$te^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(t)$	$\frac{e^{\lambda_2 t} - e^{\lambda_1 t} + (\lambda_1 - \lambda_2)te^{\lambda_1 t}}{(\lambda_1 - \lambda_2)^2}u(t)$

Let $\lambda_1 = -10 \quad \lambda_2 = -a$

Then $$y_{zs}(t) = b\,\frac{e^{-at} - e^{-10t} + (a-10)te^{-10t}}{(a-10)^2}$$

2.3) Combining 2.1 + 2.2 we get

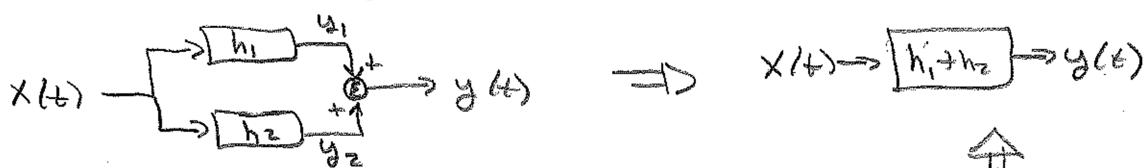
$$y(t) = Ae^{-at}u(t) + \frac{b}{(a-10^2}\left[e^{-at} - e^{-10t} + (a-10)e^{-10t}\right]u(t)$$

* The Final Result.

- As systems become larger it becomes easier to analyze (and design) them by dividing them into blocks.

Lets look at some simple cases for now (see also lecture 21)

- Parallel Blocks:

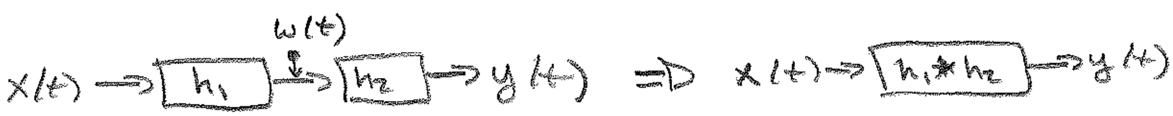


$y_1 = h_1 * x$

$y_2 = h_2 * x$

$$\Rightarrow y = y_1 + y_2 = h_1 * x + h_2 * x = (h_1 + h_2) * x$$

- Series Blocks:



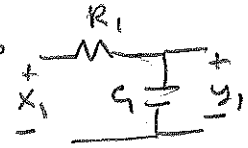
$w(t) = (h_1 * x)(t)$

$y(t) = (h_2 * w)(t)$

$$\Rightarrow y(t) = h_2 * (h_1 * x) \Rightarrow y = (h_1 * h_2) * x$$

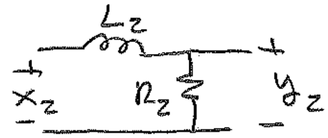
* It is important to note in block connections we ignore coupling.

Ex. Suppose System 1 was



$x_1 \to \boxed{S_1} \to y_1$

AND System 2 was

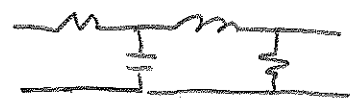


$x_2 \to \boxed{S_2} \to y_2$

When connected in series a buffer (voltage follower) must be placed in between.

$x_1 \to \boxed{} \to \boxed{} \to y_2$

corresponds to — NOT



* Another important response is the one due to the input

$x(t) = e^{st}$, the "everlasting" complex exponential
$s \in \mathbb{C}$

- The response is $y(t) = h * x = \int_{-\infty}^{\infty} h(\tau) x(t-\tau)\, d\tau$

$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)}\, d\tau = \int_{-\infty}^{\infty} h(\tau) e^{st} e^{-s\tau}\, d\tau$$

$$= e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\, d\tau = H(s) e^{st} \qquad H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\, d\tau$$

- The response is then the same as the input weighted by a value H, that depends on s.

e.g. $s = 1 + j4$

$x(t) = \underline{e^{(1+j4)t}} \to \boxed{H(s)} \to H(1+j4)\, \underline{e^{(1+j4)t}}$

Later we will see $H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\, d\tau$ is the Bilateral Laplace Transform.