Lecture #11

Convolution, and Total Response

- Last time we saw how to use convolution to find the zero state response, given the impulse response. But why does this 'work'?

Heuristic Derivation of Convolution

- If a system is LTI then
  \[ a x(t-t_0) \rightarrow ay(t-t_0) \]
- In particular
  \[ a S(t-t_0) \rightarrow ah(t-t_0) \]
- And let \( a = x(t_0) \)
- Recall sifting property of \( \delta \)
  \[ \int_{-\infty}^{\infty} f(t) S(t-t_0) \, dt = f(t_0) \]
- Because \( \delta \) is a linear operator
  \[ \int_{-\infty}^{\infty} x(t) S(t-t_0) \, dt \rightarrow [\square] \rightarrow \int_{-\infty}^{\infty} x(t_0) h(t-t_0) \, dt \]
- Now let's do a change of variables \( t = t - t_0 \) or \( t_0 = t - \tau \) and \( dt = d\tau \)
  \[ \int_{-\infty}^{\infty} x(t-t_0) S(t-t_0) \, dt \rightarrow [\square] \rightarrow \int_{-\infty}^{\infty} x(t-\tau) h(\tau) \, d\tau \]
- Again, by sifting property
  \[ x(t) \Rightarrow \square \Rightarrow \int_{-\infty}^{\infty} x(t-\tau) h(\tau) \, d\tau = h * x \]
  \[ \text{The convolution integral.} \]

Summary of time domain analysis,

- To find the total response \( y(t) \) to a system with initial conditions and input \( x(t) \),
  \[ y(t) = y_0(t) u(t) + y_{\text{eq}}(t) \text{ where } Q(D) y = P(D) x \]
- \( y_0(t) \) is the solution to \( Q(D) y = 0 \) subject to \( x \).
- \( y_{\text{eq}}(t) = (h * x)(t) \) where \( Q(D) h = S(t) \)
- or \( Q(D) h = 0 \) subject to special initial conditions.
Consider the first order filter. 

\[ V_c(t) = \frac{V_e(t)}{R_1} \]

where \( V_c(t) \neq 0 \)

and \( x(t) = e^{-10t} \) \( u(t) \)

\[ X(t) = e(t) \]

Find the total response \( y(t) \).

1. Find the Differential Equation description of system.
   1.1) KCL @ \( V_c \): \( \frac{V_c - V_c}{R_1} = CV_c' \) \( \Rightarrow \) \( V_c' + \frac{1}{R_1 C} V_c = \frac{1}{R_1 C} x(t) \)
   1.2) KCL @ \( V_\text{out} \): \( \frac{V_\text{out} - V_\text{in}}{R_2} = \frac{V_\text{in}}{R_3} \) \( \Rightarrow \) \( y(t) = \left(1 + \frac{R_2}{R_3}\right) V_\text{in}(t) \)

2. O\text{P}-\text{Amp}: \( V_c = V_\text{in} \)

Combine (1)(2)(3) \( \Rightarrow \)

\[ y'(t) + \frac{1}{R_1 C} y(t) = \left(1 + \frac{R_2}{R_3}\right) \frac{x(t)}{R_1 C} \]

\[ y(t) = A e^{-a t} + b x \]

2. The total response \( y(t) = y(t) u(t) + y_\text{dc}(t) \)

2.1) Find \( y_\text{dc}(t) \) from \( y'(t) + ay(t) = 0 \) with \( y(0^-) = A \)

\[ \alpha \left( D \right) = \alpha + \alpha \text{ has one root } -\alpha \text{ so:} \]

\[ y(t) = C e^{-\alpha t} \]

\[ y(0^-) = C = A \]

\[ y(t) = A e^{-\alpha t} \]

2.2) Find \( y_\text{dc}(t) \) from \( y'(t) + ay(t) = b x = b t e^{-10t} \)

2.2.1) Find \( h(t) \) from \( y'(t) + ay(t) = 0 \) \( y(0^+) = 1 \)

\[ y(t) = C e^{-\alpha t} \]

\[ y(0^+) = C = 1 \]

\[ h(t) = b t e^{-10t} + \left( p(t) y(t) + u(t) \right) \]

\[ p(t) = b \]

\[ = b e^{-10t} u(t) \]
Find \( \gamma_{25}(t) = (h_1 * x)(t) \)

\[ h(t) = b e^{-at} \quad x(t) = te^{ut} \]

From Table 2.1 No 9.

\[
\begin{array}{c|c|c}
X_1 & X_2 & X_1 \neq X_2 \\
- \lambda_1 (t) + u(t) & e^{\lambda_2 t} (u(t)) & \frac{e^{\lambda_1 t} - e^{\lambda_2 t} + (\lambda_1 - \lambda_2) t e^{\lambda_1 t}}{(\lambda_1 - \lambda_2)^2} u(t)
\end{array}
\]

Let \( \lambda_1 = -10 \quad \lambda_2 = -a \)

\[-at - 10t \quad -at + 10t \]

Then \( \gamma_{25}(t) = b \cdot \frac{e^{-at} - e^{-at} + (a-10)te^{-at}}{(a-10)^2} \]

2.3) Combining 2.1 + 2.2 we get

\[ y(t) = A e^{-at} u(t) + b \frac{e^{-at} - e^{-at} + (a-10)te^{-at}}{(a-10)^2} u(t) \]

The Final Result.

As systems become larger it becomes easier to analyze (and design) them by dividing them into blocks.

Let's look at some simple cases for now (see also Lecture 21)

- **Parallel Blocks**

\[ x(t) \xrightarrow{h_1} y_1(t) \quad \xrightarrow{h_2} y_2(t) \quad \Rightarrow \quad x(t) \rightarrow [h_1 + h_2] \rightarrow y(t) \]

\[ y_1 = h_1 * x \quad \Rightarrow \quad y = y_1 + y_2 = h_1 * x + h_2 * x = (h_1 + h_2) * x \]

\[ y_2 = h_2 * x \]

- **Series Blocks**

\[ w(t) \xrightarrow{h_1} y_1(t) \quad \xrightarrow{h_2} y(t) \quad \Rightarrow \quad x(t) \rightarrow [h_1 * h_2] \rightarrow y(t) \]

\[ w(t) = (h_1 * x)(t) \Rightarrow y(t) = h_2 * (h_1 * x) \Rightarrow y = (h_1 * h_2) * x \]

\[ y(t) = (h_2 * w)(t) \]
It is important to note in block connections we ignore coupling.

Ex. Suppose System 1 was

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\[ x_1 \rightarrow [S_1] \rightarrow y_1 \]
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And System 2 was

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\[ x_2 \rightarrow [S_2] \rightarrow y_2 \]
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When connected in series a buffer (Voltage Buffer) must be placed in between.

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x_1 \rightarrow [ ] \rightarrow [ ] \rightarrow y_2
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corresponds to

```
[ ] \rightarrow [ ]
```

```* Another important response is the one due to the
input
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\[ x(t) = e^{st} \]

The response is

\[ y(t) = h * x = \int h(\tau)x(t-\tau) \, d\tau \]

\[ y(t) = \int h(\tau)e^{s(t-\tau)} \, d\tau = \int h(\tau)e^{-s\tau} \, d\tau \]

\[ = e^{st}\int h(\tau)e^{-s\tau} \, d\tau = H(s)e^{st} \]

\[ H(s) = \int h(\tau)e^{-s\tau} \, d\tau \]

The response is then the same as the input weighted by a value \( H \), that depends on \( s \).

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\[ \text{e.g., } S = 1 + j4 \]

\[ x(t) = e^{(1+j4)t} \rightarrow [H(s)] \rightarrow H(1+j4)e^{(1+j4)t} \]

Later we will see \( H(s) = \int h(\tau)e^{-s\tau} \, d\tau \) is the \textbf{Bilateral Z-Transform}.