★ Convolution
- convolution integral
- graphical interpretation
- examples.

- Recall the reason we wanted to find the impulse response $h(t)$ was because that characterized the system, such that for any input $x(t)$

$$y_{zs}(t) = (h * x)(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$

- As before, we will look at the mechanics of this result first then briefly turn to its derivation.

- Definition: Given two functions $x_1(t), x_2(t)$, $t \in \mathbb{R}$ the convolution is an operator $*$ that maps both functions to another function via

$$(x_1 * x_2)(t) = \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau)\, d\tau$$

↑ called the convolution integral

- Some useful properties of convolution (dropping explicit (t))
  - commutivity: $x_1 * x_2 = x_2 * x_1$
  - distributivity: $x_1 * (x_2 + x_3) = (x_1 * x_2) + (x_1 * x_3)$
  - Associativity: $x_1 * (x_2 * x_3) = (x_1 * x_2) * x_3$
  - scaling: $(a x_1) * (b x_2) = ab(x_1 * x_2)$ a, b constant
  - Shifting: if $x_3 = x_1 * x_2$ then
    - $x_1(t-T) * x_2 = x_3(t-T)$
    - $x_1(t) * x_2(t-T) = x_3(t-T)$
    - $x_1(t-T_1) * x_2(t-T_2) = x_3(t-T_1-T_2)$

- if x_1 and x_2 are non-zero only over intervals T_1 + T_2 respectively

then $x_1 * x_2$ is non-zero only over the interval $T_1 + T_2$.

- Note importantly convolution is a linear operator.

$$(x_1 * h) = y_1 \qquad (x_2 * h) = y_2 \quad \text{then}$$

$$(a x_1 + b x_2) * h = a(x_1 * h) + b(x_2 * h) = a y_1 + b y_2$$

- Also note convolution with the δ function is the identity operator for convolution.

$$\delta(t) * x(t) = \int_{-\infty}^{\infty} \delta(\tau) x(t-\tau)\, d\tau = x(t) \quad \text{by sifting theorem.}$$

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* Graphical interpretation

- if a system is causal then $h(t) = 0$ for $t<0$ if the input is also causal then $x(t)=0$ for $t<0$ and convolution becomes

$$y_{zs}(t) = \int_{0^-}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

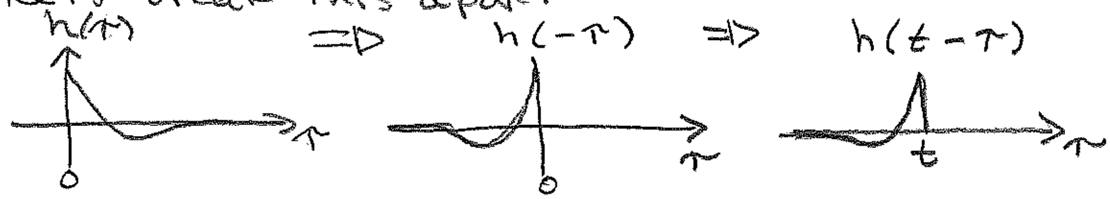
which has an intuitive graphical interpretation.

- Consider the following Example:

$$h(t) = (2e^{-2t} - e^{-t})u(t) \qquad x(t) = u(t)$$

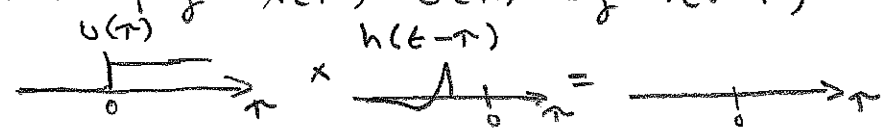
The step response is $y = \int_{0^-}^{t} x(\tau)\, h(t-\tau)\, d\tau$

- Lets break this apart

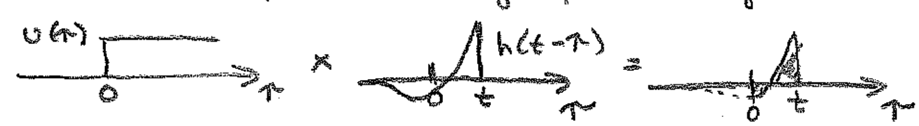


- We now multiply $x(\tau) = u(\tau)$ by $h(t-\tau)$

Ⓐ for $t<0$



Ⓑ for $t>0$



- Then we integrate (area under curve)
  for Ⓐ $t<0$ the area is zero

$\boxed{\text{DEMO}}$

For Ⓑ $t>0$ the area is some number $y(t)$

- Now lets repeat that same example doing the integrations.

$$h(\tau) = (2e^{-2\tau} - e^{-\tau})u(\tau)$$

flip $h(-\tau) = (2e^{2\tau} - e^{\tau})u(-\tau)$

shift $h(t-\tau) = (2e^{-2(t-\tau)} - e^{-(t-\tau)})u(t-\tau)$

$$(x*h)(t) = \int_{0^-}^{t} \underbrace{x(\tau)}_{u(\tau)=1} h(t-\tau)\,d\tau$$

$$= \int_{0^-}^{t} 2e^{-2(t-\tau)} - e^{-(t-\tau)}\,d\tau$$

$$= 2\int_{0^-}^{t} e^{-2(t-\tau)}\,d\tau - \int_{0^-}^{t} e^{-(t-\tau)}\,d\tau$$

$$= 2e^{-2t}\int_{0^-}^{t} e^{2\tau}\,d\tau - e^{-t}\int_{0}^{t} e^{\tau}\,d\tau$$

$$= 2e^{-2t}\left[\frac{1}{2}e^{2\tau}\Big/_{0^-}^{t}\right] - e^{-t}\left[e^{\tau}\Big/_{0^-}^{t}\right]$$

$$= 2e^{-2t}\left(\frac{1}{2}e^{2t} - \frac{1}{2}\right) - e^{-t}(e^{t} - 1)$$

$$= (1 - e^{-2t}) - (1 - e^{-t})$$

$$= e^{-t} - e^{-2t} \quad \text{for } 0 < t < \infty$$

So $y_{zs}(t) = (e^{-t} - e^{-2t})u(t)$

- The integrals get tedious for many cases but
  - Use Computer algebra System
  - use Table and properties (see Table 2.1 in text)
  - Numerically do the integration (not analytic output)

TOPS

– that same example again using Table.

$x(t) = u(t) \qquad h(t) = (2e^{-2t} - e^{-t})u(t)$

$= 2e^{-2t}u(t) - e^{-t}u(t)$

$x * h = (u(t) * 2e^{-2t}u(t)) - (u(t) * e^{-t}u(t))$

$= 2(u * e^{-2t}u(t)) - (u(t) * e^{-t}u(t))$

From Table $\quad u(t) * e^{\lambda t}u(t) = \dfrac{1-e^{\lambda t}}{-\lambda}u(t)$

So $x * h = 2\left(\underbrace{\dfrac{1-e^{-2t}}{2}}_{\lambda=-2}\right)u(t) - \left(\underbrace{\dfrac{1-e^{-t}}{1}}_{\lambda=-1}\right)u(t)$

$= (1-e^{-2t})u(t) - (1-e^{-t})u(t)$

$= (e^{-t} - e^{-2t})u(t)$ the same result.

– To practice lets use the graphical approach to find

$y(t) = (x_1 * x_2)(t)$ where $x_1(t) = u(t) - u(t-2)$

and $x_2(t) = r(t-3) - r(t-4) - u(t-4)$