* Impulse Response

– Recall from last time the total response of a system is

$$y(t) = \underbrace{y_0(t)}_{\text{zero input response}} + \underbrace{y_{z\text{-}s}(t)}_{\text{zero-state response.}}$$

Where $y_0(t)$ is the solution to the homogeneous system

$$Q(D)y = 0 \quad \text{give } D^{N-1}y(0^-),\ D^{N-2}y(0^-),\ \dots\ y(0^-)$$

– Today we focus on the first step in finding the zero-state response, determining the <u>impulse response</u>

$$\delta(t) \rightarrow \boxed{} \rightarrow h(t)$$

when no initial conditions are present.

– Reminder, the impulse response and properties.

$$\delta(t) = \begin{cases} \infty & t=0 \\ 0 & \text{else} \end{cases}, \quad \int_{-\infty}^{\infty} \delta(t) = 1, \quad x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

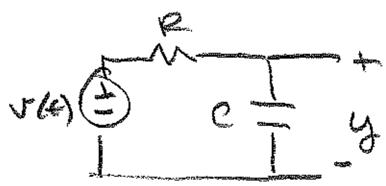
$$\frac{du}{dt} = \delta(t), \quad u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau \quad \text{AND} \quad \int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt = x(t_0)$$

– Approach #1, impulse response from step response.

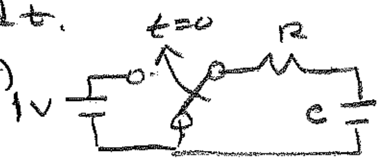
This works well when you have a physical system (e.g. circuit), that you can measure.

- First measure the step response $u(t) \rightarrow \boxed{} \rightarrow s(t)$
- Then take derivative $h(t) = \dfrac{ds(t)}{dt}$.

Example!



$v(t) = u(t) \Rightarrow$



Then $s(t) = (1 - e^{-\frac{1}{RC}t})u(t) = u(t) - e^{-\frac{1}{RC}t}u(t)$

and

$$h(t) = \frac{d}{dt}s(t) = \delta(t) - \left[\underline{e^{-\frac{1}{RC}t}\delta(t)} - \frac{1}{RC}e^{-\frac{1}{RC}t}u(t)\right]$$

$$= \delta(t) - \delta(t) + \frac{1}{RC}e^{-\frac{1}{RC}t}u(t) = \boxed{\frac{1}{RC}e^{-\frac{1}{RC}t}u(t)}$$

- When doing an analysis (as opposed to an experiment) the general case is to solve $Q(D)y = P(D)x$ when $x = \delta(t)$

- First we will just describe the procedure and get some experience, then we can see where it comes from.

Procedure to find $h(t)$, the solution to $Q(D)h(t) = P(D)\delta(t)$

1. Let $y_n(t)$ be the solution to the homogeneous system $Q(D)y_n = 0$ with auxilliary conditions $D^{N-1}y(0^+) = 1$, $D^{N-2}y(0^+) = 0$, ...rest $= 0$ ... $y(0^+) = 0$

2. Assume a form for $h(t)$

$$h(t) = \underbrace{b_0\,\delta(t)}_{\substack{=0 \text{ unless} \\ M=N}} + \underbrace{[P(D)y_n(t)]}_{\substack{\text{apply } P(D) \text{ operators} \\ \text{to solution to 1. above.}}}u(t)$$

- Example: $y' + 2y = x$ find $y(t) = h(t)$ when $x = \delta(t)$

1. $Q(D) = D + 2$ has a single root at -2

So $y_n(t) = Ce^{-2t}$ and $y(0^+) = 1$

$y_n(0^+) = C = 1$

$\therefore\ y_n(t) = e^{-2t}$

2. $P(D) = 1$, $M = 0$, $N = 1$ so $b_0 = 0$

$h(t) = 0\,\delta(t) + [1\,y_n(t)]u(t)$

$$\boxed{h(t) = e^{-2t}u(t)}$$

– Another Example: Find $h(t)$ for the system

$$y'' + 4y' + 3y = 4x$$

1. $Q(D) = D^2 + 4D + 3 = (D+1)(D+3)$

So $y_n(t) = c_1 e^{-t} + c_2 e^{-3t}$ $\quad y_n(0^+) = 0,\ y'(0^+) = 1$

$y_n(0^+) = c_1 + c_2 = 0$

$y_n'(t) = -c_1 e^{-t} - 3c_2 e^{-3t}$

$y_n'(0^+) = -c_1 - 3c_2 = 1$

Solve $c_1 + c_2 = 0 \quad -c_1 - 3c_2 = 1$

$c_1 = \frac{1}{2}\ ,\ c_2 = -\frac{1}{2}$

$\therefore\ y_n(t) = \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}$

2. $P(D) = 4 \quad m = 0,\ N = 2$ so $b_0 = 0$

$h(t) = 0\,\delta(t) + 4\,y_n(t)\,u(t)$

$= \underline{\left[2e^{-t} - 2e^{-3t}\right]u(t)}$

– Yet another Example: find $h(t)$ for the system

$$y'' + 2y' + 5y = 2x' + x$$

1. $Q(D) = D^2 + 2D + 5$ has complex roots $-1 \pm j2$

$y_n(t) = c_1 e^{(-1+j2)t} + c_2 e^{(-1-j2)t}$

or

$y_n(t) = e^{-t}\left(c_1 \cos(2t) + c_2 \sin(2t)\right)$

or

$y_n(t) = c_1 e^{-t} \cos(2t + c_2)$

– each of the 3 forms are equivalent with relationships among the constants.

– we also have that $Dy(0^+) = 1$ and $y(0^+) = 0$

- Example Cont: Let use the form

$$y_n(t) = e^{-t}(c_1 \cos(2t) + c_2 \sin(2t))$$

$$y_n'(t) = e^{-t}(-2c_1 \sin(2t) + 2c_2 \cos(2t)) + -e^{-t}(c_1 \cos(2t) + c_2 \sin(2t))$$

$$y_n(0^+) = c_1 = 0$$

$$y_n'(0^+) = 2c_2 - c_1 \Rightarrow c_2 = \frac{1}{2}$$

$$\therefore y_n(t) = \frac{1}{2} e^{-t} \sin(2t)$$

• Step 2. $P(D) = 2D + 1$ $M = 1$, $N = 2$ $b_0 = 0$

$$h(t) = 0\,\delta(t) + [(2D+1)y_n(t)]u(t)$$

$$h(t) = [2Dy_n + y_n]u(t)$$

$$h(t) = \left[2\left(\frac{-2}{2} e^{-t} \cos(2t) - \frac{1}{2} e^{-t} \sin(2t)\right) + y_n\right]u(t)$$

$$\underline{h(t) = \left[2e^{-t} \cos(2t) - \frac{1}{2} e^{-t} \sin(2t)\right]u(t)}$$

Ok, so where does this procedure come from (heuristically)?

$$D^N h + a_1 D^{N-1} h + a_2 D^{N-2} h + \cdots a_N h = \delta(t) \quad (P(D) = 1)$$

- for $t \leq 0^-$ the system is at rest and $\delta(0^-) = 0$ or $D^i h(0^-) = 0$ for $i = 0 .. N-1$.
- for $t \geq 0^+$, $\delta(0^+) = 0$ also and we have $Q(D) = 0$
- at $t = 0$, between 0^- and 0^+ the $\delta(t)$ "sets" some initial conditions at 0^+.
- integrating the above between 0^- to 0^+

$$\int_{0^-}^{0^+} D^N h + \int_{0^-}^{0^+} a_1 D^{N-1} h + \cdots \int_{0^-}^{0^+} a_N h = \int_{0^-}^{0^+} \delta(t) = 1$$

$$D^{N-1} h \Big|_{0^-}^{0^+} + a_1 D^{N-2} h \Big|_{0^-}^{0^+} + \cdots \int_{0^-}^{0^+} a_N h = 1$$

$$\underbrace{D^{N-1} h(0^+)}_{=1} + a_1 \underbrace{D^{N-2} h(0^+)}_{\text{rest} = 0} + \cdots \underbrace{\int_{0^-}^{0^+} h}_{=0} = 1$$

This is the <u>simplest</u> choice to make the equation = 1.