Impulse Response h[n]: 2.3

Recall from last time the total solution to the output of a CT-LTI system is

\[ y(t) = y_2(t) + y_{2s}(t) \]

\[ \uparrow \quad \uparrow \]

zero-to-input zero-state response

Today we focus on the first step in finding the zero-state response, determining the impulse response: \( h(t) \)

\[ \delta(t) \rightarrow \boxed{\text{?}} \rightarrow h(t) \]

If a system is LTI then in general,

\[ a \, x(t-d) \rightarrow \boxed{\text{?}} \rightarrow a \, y(t-d) \]

for constants \( a \in \mathbb{C} \) \( d \in \mathbb{R} \)

In particular,

\[ a \, \delta(t-d) \rightarrow \boxed{\text{?}} \rightarrow a \, h(t-d) \]

Constant

Now recall from the sifting property \( \delta(t) = \int_{-\infty}^{\infty} \delta(t-\tau) \, d\tau \)

Thus for the input \( x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) \, d\tau \)

So,

\[ x(t) \delta(t-d) \rightarrow \boxed{\text{?}} \rightarrow x(t) h(t-d) \]

Since integration is a linear operator,

\[ \int_{-\infty}^{\infty} x(t) \delta(t-d) \, dt \rightarrow \boxed{\text{?}} \rightarrow \int_{-\infty}^{\infty} x(t) h(t-d) \, dt \]

by sifting property

\[ x(t) \]

\[ (x * h)(t) \]

the convolution integral.
Example: finding impulse response of RC circuit.

\[ V(t) \quad \text{(source)} \quad R \quad C \quad y(t) \quad \gamma(t) \]

Who is the step response?
What is the derivative of the step response?

For the general case, given a linear ODE

\[ Q(D) y(t) = P(D) x(t) \]

we want to find the impulse response \( h(t) = y(t) \)

\[ \delta(t) = x(t) \]

Let's start with the case of \( P(D) = 1 \)

\[ Q(D) h(t) = \frac{d^n h}{dt^n} + a_1 \frac{d^{n-1} h}{dt^{n-1}} + \ldots + a_n h(t) = \delta(t) \]

If we had the auxiliary condition \( h(0^+) \) we could solve this using the homogeneous form of the DE since \( \delta(t) = 0 \) for \( t < 0 \).

Find roots of the characteristic equation \( Q(D) = 0 \)
to obtain the modes

- For real roots \( r \), the modes are \( e^{rt} \)
- For complex roots, which always occur in pairs, \( r = \alpha \pm j\omega \)
  \[ e^{\alpha t} \cos(\omega t) + e^{\alpha t} \sin(\omega t) \]

The solution is then the sum of these modes.

\[ y_h(t) = \sum A e^{rt} + \ldots + e^{\alpha t} (B \cos(\omega t) + C \sin(\omega t)) \]

For complex roots \( \alpha \pm j\omega \)
Now if we had \( h(0^+) \) then we could solve for the constants.

- For \( t < 0 \), \( \delta(t) = 0 \) and \( h(t) = 0 \) (zero-state)

- For \( t > 0 \), \( \frac{d^n h}{dt^n} + a_1 \frac{d^{n-1} h}{dt^{n-1}} + \ldots + a_0 h(t) = \delta(t) \)

In order for the two sides to be equal, the LHS must contain a \( \delta(t) \), but it can't be \( h(t) \) because then the RHS would have a derivative of the \( \delta \) (called a doublet).

Thus, \( h(t) \) is discontinuous at \( t = 0 \), but no \( \delta \) function appears in the expression for \( h \). Thus \( \int_{0^-}^{0^+} h(t) \, dt = 0 \)

Integrating the above from \( 0^- \) to \( 0^+ \) gives

\[
\int_{0^-}^{0^+} \frac{d^n h}{dt^n} \, dt + a_1 \int_{0^-}^{0^+} \frac{d^{n-1} h}{dt^{n-1}} \, dt + \ldots + a_0 \int_{0^-}^{0^+} h(t) \, dt = \int_{0^-}^{0^+} \delta(t) \, dt = \begin{cases} 1 & \text{by definition} \\
0 & \text{else} \end{cases}
\]

\[
\frac{d^{n-1} h}{dt^{n-1}} + a_1 \frac{d^{n-2} h}{dt^{n-2}} + \ldots + a_0 h(t) = 1
\]

\[
\frac{d^{n-1} h(0^+)}{dt^{n-1}} + a_1 \frac{d^{n-2} h(0^+)}{dt^{n-2}} + \ldots + a_0 h(0^+) = 1
\]

There are several choices of auxiliary conditions that make the above expression true. By convention, we will use \( \frac{d^{n-1} h(0^+)}{dt^{n-1}} = 1, \text{ rest } = 0 \).
Finally if \( h(t) \) is a solution to
\[
Q(D) h(t) = S(t)
\]
then by linearity \( P(D) h(t) \) is a solution to
\[
Q(D) h(t) = P(D) S(t)
\]
except a special case, where \( M = N \) the RHS of \( P(D) S(t) \)
does have an \( N \)th derivative of \( S(t) \) thus \( h(t) \) must
have a \( S(t) \) in it as well.

This all leads us to the following procedure.

Given \( Q(D) y(t) = P(D) x(t) \) with \( x(t) = S(t) \)

1. Let \( y_n(t) \) be the homogeneous solution of
   \[
   Q(D) y_n(t) = 0
   \]
   with auxiliary conditions
   \[
   \frac{d^{N-1} y_n(0^+)}{dt^{N-1}} = D \quad y_n(0^+) = 1
   \]
   \[
   D^{N-2} y_n(0^+) = 0
   \]
   \[
   \vdots
   \]
   \[
   D^{N-N} y_n(0^+) = 0
   \]
   \[
   \text{where } D = 0.
   
   2. Assume a form for \( h(t) \)

\[
0 \text{ unless } M > N
\]

\[
= D \quad \downarrow
\]

\[
h(t) = b_0 S(t) + \left[ P(D) y_n(t) \right] u(t)
\]

\[
\text{apply operators (derivatives)}
\]

\[
to \ y_n(t)
\]