* Zero-input response

- Last time we saw that linear, constant-coefficient differential equations were the tool of choice for working with our systems of interest.

LCCDE
- Internal Form: $\dot{\underline{z}} = A\underline{z} + Bx$, $y = C\underline{z} + Dx$ state equation
- External Form:

$$\frac{d^N y}{dt^N} + a_1 \frac{d^{N-1} y}{dt^{N-1}} + a_2 \frac{d^{N-2} y}{dt^{N-2}} + \cdots a_N y = x$$

which is the form we will work with.

- Two important remarks

1. LCCDE are equivalent to LTI systems only when the auxillary conditions are zero.
2. The input may also have a differential form

$$\frac{d^N y}{dt^N} + a_1 \frac{d^{N-1} y}{dt^{N-1}} + \cdots a_N y = b_{N-M} \frac{d^M x}{dt^M} + \cdots b_N x$$

N = order of Left-hand side
M = order of Righ-hand side

$N > M$ in practice.

This is the general case we will look at.

- It is helpful to clean up that notation using the <u>derivative operator</u>

$$D^n = \frac{d^n}{dt^n} \qquad D^2 x = \frac{d^2 x}{dt^2} \text{ for example}$$

$$D^0 y = y$$

To give:

$$D^N y + a_1 D^{N-1} y + \cdots a_N y = b_{N-M} D^M x + b_{N-M-1} D^{M-1} x + \cdots b_N x$$

we can factor out the operators.

$$\underbrace{(D^N + a_1 D^{N-1} + \cdots a_N)}_{\substack{\text{polynomial in } D \\ Q(D)}} y = \underbrace{(b_{N-M} D^M + b_{N-M-1} D^{M-1} + \cdots b_N)}_{\substack{\text{polynomial in } D \\ P(D)}} x$$

To Give the compact form: $Q(D)y = P(D)$ for the external description.

- That notation can seem a bit daunting at first, lets do some examples.

  - Given $y'' + 3y' + 6y = 3x$ in the compact form?
  - Given $\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 6y = 3\frac{d^2x}{dt} + \frac{dx}{dt} + 5x$

    write it in the compact form.

★ Solving $Q(D)y = P(D)x$

- Recall from your differential Equations course that the procedure to solve LCCDE was to split it into the homogeneous solution (aka, the natural response) $y_h(t)$ and the particular solution (aka, the forced response) $y_p(t)$

  The total solution was $y(t) = y_h(t) + y_p(t)$

  Given the auxillary conditions:

  $$D^{N-1}y(t_0),\ D^{N-1}y(t_0),\ \dots\ y(t_0) \text{ for some } t_0 \geq 0$$

  AND an input.

- the solution to the homogeneous problem is

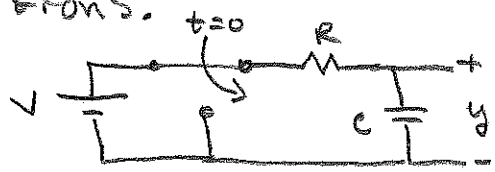
  $$Q(D)\,y = 0 \quad \text{Given auxillary conditions.}$$

  where $y(t) = \sum_{i=1}^{N} C_i e^{\lambda_i t}$ where λ_i is the ith root of the characteristic equation $P(D) = 0$

  C_i: constant; $e^{\lambda_i t}$: characteristic modes.

  solving for the constants using the auxillary conditions.

- Example:



$y' + ay = ax \quad a = \frac{1}{RC}$

$x = V u(-t)$

For $t < 0$

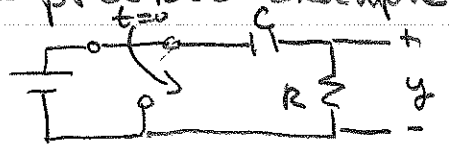
$y(0^-) = V$

For $t = 0^+$

$y(0^+) = V$

$y(0^-) = y(0^+) = V$

- What if I reverse the order of the components in the previous example.



$C V_c' = \frac{y}{R}$ $\quad V_c = x - y$

$V_c' = x' - y'$

$(x' - y') = \frac{y}{RC}$

or

$y' + \frac{1}{RC} y = x'$ $\quad$ Let $a = \frac{1}{RC}$

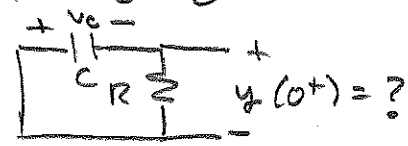
$b_0 = 1$

To solve, $y_h(t) = C e^{-at}$

we need $y(0^+)$

For $t < 0$

But for $t = 0^+$



Ohms Law says.

$y(0^+) = R \underbrace{C V_c'(0^+)}_{i(0^+)} = V_c(0^+)$

This says that $y(0^+) = V_c(0^+) = V$ which requires an impulse of current from 0^- to 0^+.

- This is an odd case, but highlights the fact we want to use a specific auxillary condition, the initial condition at $t = 0^-$, even though this is outside the usual DiffEq solution domain.

- Another difference between our approach and normal LCCDE is we will consider arbitrary inputs, and arbitrary order
  - Recall the way one solves for the particular solution to a second order LCCDE

1. method of undetermined coefficients
   - guess solution based on form of input
     e.g $x(t) = \cos(\omega t)$ $\quad y_p(t) = A\cos(\omega t) + B\sin(\omega t)$

     solving for A, B by equating coefficients

2. method of variation of parameters.
   - use homogeneous solutions $y_1(t)$ + $y_2(t)$ and

     $y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$

     where $u_1' y_1 + u_2' y_2 = 0$

     $u_1' y_1' + u_2' y_2' = k(t)$

Neither work for arbitrary X

- So, we will take a different approach.

$$y(t) = y_0(t)u(t) + y_{zs}(t)$$

- where $y_0(t)$ is the <u>zero-input</u> response, the solution to $Q(D)y_0 = 0$ using the 0^- initial conditions $D^{N-1}y(0^-), D^{N-2}y(0^-), \ldots y(0^-)$
- y_{z-s} is the <u>zero-state</u> response, when we assume $D^{N-1}y(0^-) = 0$, $D^{N-2}y(0^-) = 0$, $\ldots$ $y(0^-) = 0$.

- Today, we are only interested in the zero-input response.
  - This is almost identical to the homogeneous solution just with different auxillary conditions.
    - Given $Q(D)y = 0$ the solution is

$$y_0(t) = \sum_{i=1}^{N} c_i e^{\lambda_i t}$$

where c_i are constants and λ_i are roots of $Q(D) = 0$.

- λ's are simple roots (real)
- λ's are in complex pairs

- Using our example from before:

$$y' + ay = ax \quad \underline{\text{or}} \quad y' + ay = x'$$

the zero-input equation is the same $y' + ay = 0$

$Q(D) = 0 \Rightarrow D + a = 0$ or $\lambda = -a$

Thus $y_0(t) = Ce^{-at}$ where $a = \frac{1}{RC}$

- C is determined from $y(0^-) = V = Ce^{-a(0)} = C$

$\therefore \; y_0(t) = Ve^{-\frac{1}{RC}t}$

- An example with complex roots.

$$y'' + 2y' + 2y = x \qquad y(0^-) = 1 \qquad y'(0^-) = 0$$

roots of $Q(D) = D^2 + 2D + 2 = 0$ are $\lambda = -1 \pm j$

So $y_0(t) = c_1 e^{(-1+j)t} + c_2 e^{(-1-j)t}$ form (1)

or using Eulers identities

$y_0(t) = e^{-t}(c_1 \cos(t) + c_2 \sin(t))$ form (2)

or using trig identity

$y_0(t) = c_1 e^{-t} \cos(t + c_2)$ form (3)

- forms (1), (2), (3) are equivalent

• using form (2) solve for constants.

$$y_0(t) = c_1 e^{-t} \cos(t) + c_2 e^{-t} \sin(t)$$

$$y_0'(t) = -c_1 e^{-t} \sin(t) + -c_1 e^{-t} \cos(t) + c_2 e^{-t} \cos(t) + -c_2 e^{-t} \sin(t)$$

$$y(0^-) = c_1(1) + c_2(0) = 1$$

$$y'(0^-) = 0 - c_1 + c_2 + 0 = 0 \Rightarrow c_2 = c_1 = 1$$

$$\therefore \quad y_0(t) = e^{-t}(\cos(t) + \sin(t))$$