Zero-Input Response

Laithi 2.1 & 2.2

- time domain analysis
- zero-input response

This class covers three techniques for analysis of signals & systems.

Tool of choice: X(t) -> ____________________ -> y(t)  

"Time Domain" analysis
The domain of the functions is t ∈ \( \mathbb{R} \).

The tool of choice is algebra

\[ X(s) \rightarrow \text{[ ]} \rightarrow Y(s) \]

"Laplace Domain" analysis
The domain of the functions are \( s \in \mathbb{C} \).

The tool of choice is algebra

\[ X(\omega) \rightarrow \text{[ ]} \rightarrow Y(\omega) \]

"Fourier Domain" analysis
The domain of the functions are \( \omega \in \mathbb{R} \).

Time domain analysis of Linear Time-Invariant Continuous Systems (LTICT) is equivalent to solving linear constant coefficient differential Equations.

Define the operator

\[ D^n = \frac{d^n}{dt^n} \begin{bmatrix} \end{bmatrix} \]

The general form is then:

\[ \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_N y = b_{n-m} \frac{d^m x}{dt^m} + b_{n-m-1} \frac{d^{m-1} x}{dt^{m-1}} + \ldots + b_0 x \]

or in operator form

\[ (D^n + a_1 D^{n-1} + \ldots + a_N) y = (b_{n-m} D^m + b_{n-m-1} D^{m-1} + \ldots + b_0) x \]
So \( N \) = order of the LHS \( M \) and of RHS \( N > M \) in practice (we will see why later)

The LHS polynomial of operators we denote \( Q(D) \)

The RHS polynomial of operators we denote \( P(D) \)

Then we have \( Q(D) y(t) = P(D) x(t) \)

Recall from your differential equations course the method used to solve linear constant coefficient DEs was to split \( P(D) \) into the homogeneous solution (aka natural response) \( y_n(t) \) and the particular solution (aka forced response) \( y_f(t) \).

The total response \( y(t) = y_n(t) + y_f(t) \)

The natural response is the solution of \( Q(D)y_n(t) = 0 \)

which consists of a linear combination of the characteristic modes, the roots of the characteristic Eq \( Q(D) = 0 \). The constants of the solution are determined from the auxiliary conditions \( y(0^+), y'(0^+), y''(0^+) \) etc.

E.g.

\[ y' + ay = ax \quad a = \frac{1}{RC} \]

\[ Q(D) = D + a \quad y_n(t) = Ce^{-at} \quad t \geq 0^+ \]

\[ y_n(t) = Ce^{-at} \quad t \geq 0^+ \]

need \( y(0^+) \)

\[ \text{For } t < 0 \]

\[ \text{For } t > 0 \]

\[ y(0^-) = V \]

\[ y(0^+) = V \]

\[ \frac{dv_n(t)}{dt} = \frac{V}{RC} \]

\[ \text{Voltage cannot change instantaneously} \]

\[ \text{thus} \]
E.g. continued... what if I reverse the components

\[ C_{Vc}' = \frac{y'}{R} \quad V_c = x - y \]
\[ V_c' = x' - y' \]
\[ (x' - y') = \frac{y}{RC} \]
\[ y' + \frac{1}{RC} y = x' \quad \text{let } a = \frac{1}{RC} \]

\[ Q(D) = D + a \]

\[ y_n(t) = Ce^{-at} \]

need \( y(0^+) \)

- For \( t < 0 \)

\[ y(0^-) = 0 \quad \frac{1}{R} \int_{0^-}^{0^+} y_c \, dt = 0 \]
\[ V_c(0^-) = V \]

- Conclusion: it is easier for us to find initial conditions at \( 0^- \) than at \( 0^+ \) (auxiliary conditions)

- So what about the forced response? The most common solution method is the "method of undetermined coefficients."

Given an input with a finite number of independent derivatives, there is a generic forced response, e.g.,

\[ x(t) = \cos (\omega t) \]
\[ y_f(t) = A \cos (\omega t) \]

where the constant is found by equating coefficients

\[ Q(D)y_f(t) = P(D)x(t) \]

- Conclusion: this method does not work for every input.
For these reasons the classical methods from DE don't work very well for us. So we will use a different expansion of the solution:

\[ y(t) = y_0(t) + y_{zs}(t) \]

\( y_0(t) \) is the zero-input response, the solution of
\[ Q(D)y_0(t) = 0 \quad \text{given } y(0^-), y'(0^-) \quad \text{ auxiliary conditions} \]

\( y_{zs}(t) \) is the zero-state response, the solution of
\[ Q(D)y_{zs}(t) = p(D)x(t) \quad \text{with } y(0^-) = 0, y'(0^-) = 0 \quad \text{ etc.} \]

Example: Given the following circuit, what is the zero-input response.

Where \( V_c(0^-) = V \).