★ LTI systems described by linear, constant coeff, DE's.

- Last time we classified systems and limited ourselves to a class of systems for the remainder.

  Linear - Time-Invariant (LTI) Systems.

- Today we will see how LCCDE's provide the natural framework for studying LTI systems from an external perspective.

★ Internal v/s External descriptions.

- Consider a first-order system with input/output relationship

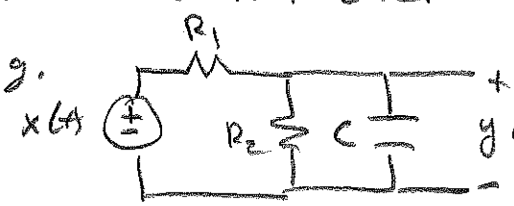
  causal $x(t) \longrightarrow \boxed{} \longrightarrow y(t) = e^{-at}\int b e^{a\tau} x(\tau)\,d\tau$

  This is an <u>external</u> description of the system.

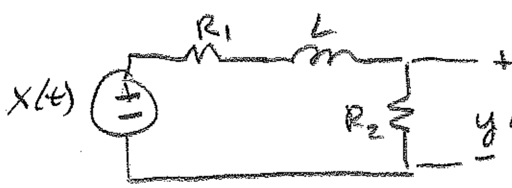
- external system descriptions tell us nothing about the details inside the block.

- external descriptions are not unique to a system. multiple <u>internal</u> descriptions could lead to the same external one.

E.g.



$$\Rightarrow y' + \underbrace{\frac{R_1+R_2}{R_1R_2C}}_{a} y = \underbrace{\frac{1}{R_1C}}_{b} x \quad \text{above}$$



$$\Rightarrow y' + \underbrace{\frac{R_1+R_2}{L}}_{a} y = \underbrace{\frac{R_2}{L}}_{b} x \quad \text{above}$$

From an external perspective, for different component values with same a, b the two systems appear identical.

Both are described by a 1^{st} order LCCDE.

Note: # energy storage elements = # internal states in circuits.

Question: If LTI systems are just LCCDE why have a whole other course?

Answers: – for non homogeneous DE you were likely only taught 1^{st} & 2^{nd} order with a couple of standard inputs.

– Method of undetermined coefficients
– Method of variation of parameters.

But we are interested in systems with arbitrary order and input

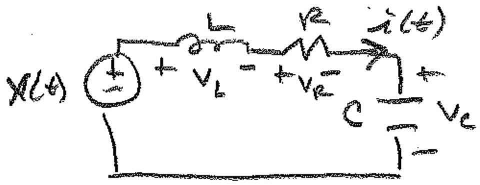
You may or may not have learned these in DiffEq. ⟹ This requires use of <u>convolution</u> or <u>Laplace</u> to solve for causal inputs.

For non-causal inputs we need <u>Bilateral</u> Laplace or Fourier techniques.

Exactly which depends on periodic or not.

✱ Internal descriptions

Lets look at a 2^{nd} order example.



– 2 energy storage elements so 2 states

1. current through inductor

2. Voltage across capacitor.

Analysis: (1) KVL: $V_L + V_R + V_c = x$

(2) Relationship between V_L and i is $V_L = L\frac{di}{dt}$

(3) put (2) ⟹ (1) solve for $\frac{di}{dt} = -\frac{R}{L}i - \frac{1}{L}V_c + \frac{1}{L}x$ (5)

State equation ⟹ derivative of state var; in terms of other states; and input.

(4) Relationship between V_c & i: $i = C\frac{dV_c}{dt}$ or $\frac{dV_c}{dt} = \frac{1}{C}i$

another state equation

- Lets combine the state equation (3) + (4) into matrix-vector form.

$$\begin{bmatrix} \frac{di}{dt} \\ \frac{dv_c}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i \\ v_c \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} x.$$

derivatives of states i & v_c.

- Let $z(t) = \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix}$ be the <u>state vector</u>

a collection of signals (states)

$z_1 = i \quad z_2 = v_c$

Then $\frac{dz}{dt} = \underset{\substack{\uparrow \\ 2\times 2 \\ \text{matrix}}}{A} z + \underset{\substack{\uparrow \\ \text{vector}}}{B} x$ • This is a 2D <u>linear</u> state equation.

* This state equation is an <u>internal</u> description * of the system because z_1 + z_2 correspond to physically interpretable states.

Solving it gives us the time evolution of the states. $z(t)$

- Now suppose we wanted to treat the output as the resistor voltage v_R

$y = v_R = R i = R z_1(t)$ is a simple (non-dynamic) function of states.

We call this the <u>output</u> <u>equation</u>.

- In summary an internal description is a state equation and output equation where the states have a physical correspondace to the system.

$$\dot{z} = A z + B x$$
$$y = C z + D x$$

for matrix A, C and vector C, D constants.

Note: How to solve these directly in this form is outside the scope of the course.

★ external description.

- Going back to our circuit analysis we had

(3) $\frac{di}{dt} = -\frac{R}{L} i - \frac{1}{L} v_c + \frac{1}{L} x$

(4) $\frac{dv_c}{dt} = \frac{1}{C} i$

- Suppose we are interested in the cap voltage as the output, $y = V_c$

Then if we take the derivative of (4) and substitute it and (4) into (3) we get.

$$y'' + \frac{R}{L} y' + \frac{1}{LC} y = \frac{1}{LC} x$$

A 2^{nd} order LCCDE in y. This is an external description.

- compare this to the internal description

$$\begin{bmatrix} \dot{i} \\ \dot{V}_c \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i \\ V_c \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} x$$

with output equation

$$y = V_c$$

- we loose the evolution of the state i (con)
- we would have to rederive & resolve the external LCCDE for each new output (con)
- But, you already know how to solve the external case (for some "standard" inputs) (pro)

Summary: - internal descriptions are more detailed but (conceptually) harder to solve
- external descriptions are easier to solve but loose detail.

- we will focus on external descriptions.

- They agree (almost, see stability lecture) most of time.