Classifying Systems

- linear / non-linear
- time invariant / time varying
- memory less / dynamic
- causal / non-causal
- invertible / non-invertible
- stable v/s unstable

In this course we study linear, time-invariant systems. Usually they are dynamic and causal.

A system is linear if the principle of superposition holds. If

\[ x_1(t) \rightarrow [\quad \rightarrow y_1(t) \quad \text{response to } x_1 \]

and \[ x_2(t) \rightarrow [\quad \rightarrow y_2(t) \quad \text{response to } x_2 \]

then if the system is linear and \( a, b \in \mathbb{R} \)

\[ a x_1(t) + b x_2(t) \rightarrow [\quad \rightarrow a y_1(t) + b y_2(t) \]

otherwise it is non-linear.

Example: 1st order differential equation

\[ y' + ay = x \quad \text{then the solution is} \]

\[ y(t) = e^{-at} \int e^{at} x(t) \, dt \]

\[ y_1(t) = e^{-at} \int e^{at} x_1(t) \, dt \]

\[ y_2(t) = e^{-at} \int e^{at} x_2(t) \, dt \]

If \( x(t) = ax_1(t) + bx_2(t) \)

\[ y(t) = e^{-at} \int e^{at} (ax_1(t) + bx_2(t)) \, dt \]

\[ = a e^{-at} \int e^{at} x_1(t) \, dt + b e^{-at} \int e^{at} x_2(t) \, dt \]

\[ = a y_1(t) + b y_2(t) \quad \text{so it is a linear system.} \]
An example of a non-linear system is the pendulum. Consider different initial values of $\theta$ as the input:

\[ \ddot{\theta} + \frac{g}{L} \sin \theta = 0 \]

non-linear function.

- Consider different initial values of $\theta$ as the input.

\[ \frac{d^2 \theta}{dt^2} + \frac{g}{L} \sin \theta = 0 \]

$\theta(0) = \theta_1$

$\theta'(0) = 0$ no velocity

- The response to the third case $\theta(0) = \theta_1 + \theta_2$

is not the sum of the response when $\theta(0) = \theta_1$

- However, a common tactic is to linearize. We can approximate $\sin(\theta) = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots$ around $\theta = 0$ neglect these terms

$\sin(\theta) \approx \theta$ for small $\theta$.

The DE becomes then $\ddot{\theta} + \frac{g}{L} \theta = 0$ Characteristic Eq.

With solution $\theta = C_1 \cos \left( \sqrt{\frac{g}{L}} t \right) + C_2 \sin \left( \sqrt{\frac{g}{L}} t \right)$ or $r = \pm \sqrt{\frac{g}{L}}$

- Where $C_1$, $C_2$ are determined from the initial conditions $\theta(0) + \dot{\theta}(0)$.

- This revised model is linear.

Examples show the above is a linear system.

\[ \ddot{\theta} + \frac{g}{L} \theta = 0 \]

$\theta(0)$ is traying

$\dot{\theta}(0) = 0$
A system is time invariant if the response does not depend on when the input is applied.

\[ x(t) \rightarrow [\square] \rightarrow y(t) \]
\[ x(t-T) \rightarrow [\square] \rightarrow y(t-T) \]

Any constant coefficient differential equation is time invariant.

Example: Consider our RC circuit.

\[ R \quad C \quad y \]
\[ y(0) = y_0 \]

This is a linear constant coefficient DE

\[ y' + \frac{1}{RC} y = \frac{1}{RC} x \]

as long as the values of \( R \) and \( C \) do not change.

*Note this is an approximation, real circuit elements do change over time, hopefully only a little and slowly.

A system is memory less if the output depends only on the current input, i.e., \( y(t) = F(x(t)) \).

Otherwise, the system has memory or is dynamic.

Any system with a differential equation model (that is non-trivial) is dynamic.

E.g., again using our circuit

\[ y' = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h} \]

the solution depends on \( y(t-h) \) by \( y(t) \).

Another way of seeing this is the solution.

\[ y(t) = e^{-\frac{t}{RC}} \int_0^t e^{\frac{t'}{RC}} x(t') \; dt' \]

has an integral.
- A system is causal if the response depends only on current & past values of \( x(t) \). It can tell the future.

\[
y(t_0) = F(x(t)) \quad \text{for} \quad t \leq t_0 \quad \text{only}.
\]

All real physical systems are causal.

- A system is invertible if you can obtain \( x(t) \) from \( y(t) \).

\[
x(t) \longrightarrow [\text{process}] \rightarrow y(t) \rightarrow [T^{-1}] \rightarrow x(t)
\]

This concept is useful in many areas, in particular communication:

\[
x(t) \rightarrow [\text{encode}] \rightarrow \text{Channel} \rightarrow [\text{decode}] \rightarrow x(t)
\]

And system identification:

\[
x(t) \rightarrow [\text{process}] \rightarrow y(t) \rightarrow [T^{-1}] \rightarrow x(t)
\]

Invertible \( \text{if} = T^{-1} \) I have the unknown system.

We will see later how to find system inverses for deterministic systems.

- Stability: There are 2 definitions of stability

  Internal & external (Bounded Input Bounded Output)

  \( BIBO \)

They are usually but not always equivalent.
Stability is important because any practical useful system must be stable.

Today we will define BIBO Stability. Internal stability needs to wait until we cover internal descriptions.

BIBO Stability is based on an external description.

\[ x(t) \rightarrow \text{square} \rightarrow y(t) \]

\[ \text{If the system is BIBO Stable} \]

\[ y(t) \text{ is bounded} \]

as long as \( x(t) \) is bounded

\[ \text{do not know details.} \]

* It may seem we can't "not" "know" the details, but we will see how later.

We will spend a good deal of time on determining stability because it is so important. It is also a major part of 3704.

Note BIBO Stability can depend on the space of the output / interpretation of output.

Example: An electric motor where the input is voltage and the output is the angle.

Of course \( \Theta + 2\pi n \) is equivalent to \( \Theta \) and viewed in Cartesian coordinates a point on this circle is bounded.