- some properties of periodic functions
- even/odd signals
- impulse signal } singularity functions
- step signal

- Recall the periodic functions are those for which

$$f(t) = f(t + nT_0) \quad n \in \mathbb{Z} \quad T_0 \in \mathbb{R}$$

Some helpful properties

- if $f(t)$ is periodic with period P and g is any function then $g(f(t))$ is periodic with period P.

  Example: $f(t) = \sin(2\pi t)$ $P = 1$

  $g(x) = x^2$

  $\sin^2(2\pi t)$ is periodic with $P = 1$.

- if $f(t)$ is periodic with Period P
  $g(t)$ is periodic with Period Q

  and if there exists positive integers a, b s.t. $aP = bQ = R$

  then $f(t) + g(t)$ and $f(t)g(t)$ are periodic with Period R.

  Example: $f(t) = \sin(2\pi t)$ $P = 1$

  $g(t) = \cos(10\pi t + \frac{\pi}{4})$ $Q = \frac{1}{5}$

  if $a = 2$ $b = 10$ then

  $\sin(2\pi t)\cos(10\pi t + \pi/4)$ is periodic with period $R = 2$

- The last property implies that P and Q must both be rational in π or neither.

  Example: $f(t) = \sin(\frac{\pi}{2}t + 3)$ $f = \frac{\frac{\pi}{2}}{2\pi} = \frac{1}{4}$

  $P = \frac{1}{f} = 4$

  $g(t) = \cos(5t)$ $f = \frac{5}{2\pi}$ $Q = \frac{2\pi}{5}$

  there is no $a, b \in \mathbb{Z}$ s.t. $a \cdot 4 = b\frac{2\pi}{5} = R$

* Even / ODD signals

- a signal is <u>even</u> if $f(t) = f(-t) \ \forall t$
  e.g. $\cos(t)$ is even $\sin(t)$ is not
- a signal is <u>odd</u> if $f(t) = -f(-t) \ \forall t$
  e.g. $\sin(t)$ is odd $\cos(t)$ is not.
- every signal can be written as a sum of even and odd signals

$$f(t) = f_e(t) + f_o(t) \quad \text{where}$$

$$f_e(t) = \frac{1}{2}\left[f(t) + f(-t)\right]$$

$$f_o(t) = \frac{1}{2}\left[f(t) - f(-t)\right]$$

Note: $\displaystyle\int_{-\infty}^{\infty} f_e(t)\,dt = \int_{-\infty}^{\infty} f(t)\,dt$

and

$\displaystyle\int_{-\infty}^{\infty} f_o(t)\,dt = 0$

This identity will come in handy. a few times this semester.

- Example: write $x(t) = e^{-5t^2}\left(\sin(2\pi t) + \cos(2\pi t)\right)$ as even/odd components.

$$x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right] = e^{-5t^2}\cos(2\pi t)$$

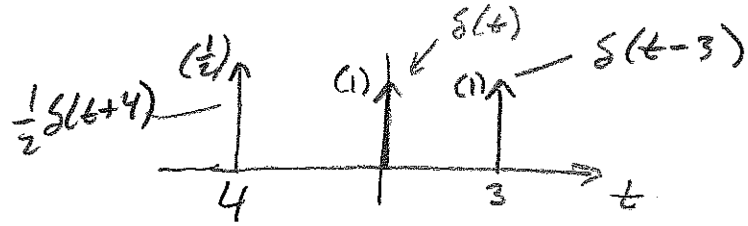
$$x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right] = e^{-5t^2}\sin(2\pi t)$$

- The above example demonstrates some properties of even/odd signals.
  - The product of two even signals is <u>even</u>
  - The product of an even and an odd signal is <u>odd</u>
  - The product of two odd signals is <u>even</u>

★ Impulse function

Another important signal is the Dirac Delta function a.k.a the <u>impulse function</u>.

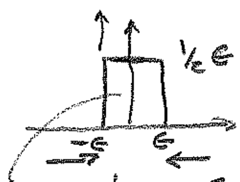
$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases} \quad \text{where} \quad \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$



The impulse models a "kick" to the system.

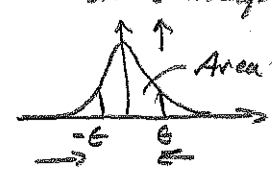
- Mathematically the impulse is not a function but a distribution. We can gain some appreciation of this by looking at some equivalent defs.

$$\delta(t) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \sqcap\left(\frac{t}{2\epsilon}\right) = \begin{cases} \frac{1}{2\epsilon} & |t| \leq \epsilon \\ 0 & \text{else} \end{cases}$$

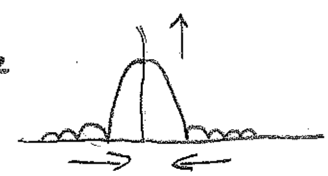


DEMO

$$\delta(t) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{2\pi}\,\epsilon} e^{-\frac{t^2}{2\epsilon^2}} \quad \text{the Gaussian function}$$



$$\delta(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\frac{\sin\left(\frac{\pi t}{\epsilon}\right)}{\frac{\pi t}{\epsilon}} \right]^2 \quad \text{the sinc}^2 \text{ function}$$



- Two useful properties of the impulse function:

  - Sampling $\quad f(t)\delta(t) = f(0)\delta(t)$

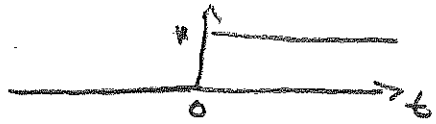
  - sifting $\quad \int_a^b f(t)\delta(t)\,dt = f(0)$ for any $a < 0 < b$

Q: would you consider $\delta(t)$ even or odd?

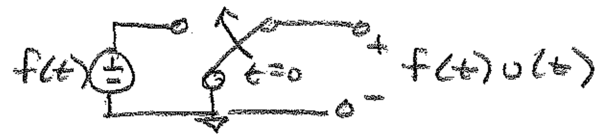
★ Unit Step or Heaviside Function.

We have already been using this function.

$$u(t) = \begin{cases} 0 & t<0 \\ 1 & t \geq 0 \end{cases} \quad \text{or} \quad u(t) = \begin{cases} 0 & t<0 \\ \frac{1}{2} & t=0 \\ 1 & t>0 \end{cases} \quad \text{or undefined}$$



u(t) models a switch



- The unit step can be defined in terms of the impulse and vice-versa

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau \quad \text{AND} \quad \delta(t) = \frac{du}{dt}$$

using the notion of distributions.

E.g.

$$\delta(t) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{2\pi}\,\epsilon} e^{-t^2/2\epsilon^2}$$

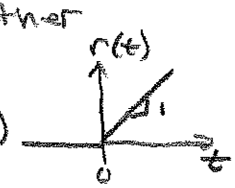
$\boxed{\text{DEMO}}$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau = \lim_{\epsilon \to 0} \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\,\epsilon} e^{-\tau^2/2\epsilon^2}\, d\tau$$

$$= \lim_{\epsilon \to 0} \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{t}{\sqrt{2}\,\epsilon}\right)\right)$$

- We can also use the unit step to define other signals.

e.g. RAMP $\quad r(t) = \int_{-\infty}^{t} u(\tau)\, d\tau = t\,u(t)$



e.g. Pulse $\quad p(t) = u(t) - u(t-\epsilon)$

a pulse of length ϵ