some properties of periodic functions
- even/odd signals
- impulse signal
- step signal
- singular value functions

Recall the periodic functions are those for which
\[ f(t) = f(t + nT_0) \quad n \in \mathbb{Z} \quad T_0 \in \mathbb{R} \]

Some helpful properties

- if \( f(t) \) is periodic with period \( P \) and \( g(t) \) is any function then \( g(f(t)) \) is periodic with period \( P \).

  Example: \( f(t) = \sin(2\pi t) \quad P = 1 \)
  \[ g(x) = x^2 \]
  \[ \sin^2(2\pi t) \text{ is periodic with } P = 1 \]

- if \( f(t) \) is periodic with period \( P \) and \( g(t) \) is periodic with period \( Q \) and there exists positive integers \( a, b \) s.t. \( aP = bQ = 1 \).

  then \( f(t) + g(t) \) and \( f(t)g(t) \) are periodic with period \( R = \frac{1}{aP} \).

  Example: \( f(t) = \sin(2\pi t) \quad P = 1 \)
  \[ g(t) = \cos(10\pi t + \frac{\pi}{4}) \quad Q = \frac{1}{5} \]
  if \( a = 2 \quad b = 10 \) then
  \[ \sin(2\pi t)\cos(10\pi t + \frac{\pi}{4}) \text{ is periodic with period } R = 2 \]

- The last property implies that \( P \) and \( Q \) must both be rational in \( \Pi \) or neither.

  Example: \( f(t) = \sin\left(\frac{\pi}{2} t + 3\right) \quad f = \frac{\pi}{2} \quad P = 1 \)
  \[ g(t) = \cos(5t) \quad f = \frac{5}{2\pi} \quad Q = \frac{2\pi}{5} \]
  there is no \( a, b \in \mathbb{Z} \) s.t. \( a \cdot \frac{1}{2} = b \cdot \frac{2\pi}{5} = R \)
Even/Odd signals

- A signal is \textit{even} if \( f(t) = f(-t) \) \( \forall t \)
  - e.g., \( \cos(t) \) is even, \( \sin(t) \) is not.

- A signal is \textit{odd} if \( f(t) = -f(-t) \) \( \forall t \)
  - e.g., \( \sin(t) \) is odd, \( \cos(t) \) is not.

- Every signal can be written as a sum of even and odd signals
  \[ f(t) = f_e(t) + f_o(t) \] where
  \[ f_e(t) = \frac{1}{2} \left[ f(t) + f(-t) \right] \]
  \[ f_o(t) = \frac{1}{2} \left[ f(t) - f(-t) \right] \]

Note: \( \int_{-\infty}^{\infty} f_e(t) \, dt = \int_{-\infty}^{\infty} f(t) \, dt \)
and
\( \int_{-\infty}^{\infty} f_o(t) \, dt = 0 \)
This identity will come in handy, a few times this semester.

- Example: write \( x(t) = e^{-5t^2} (\sin(2\pi t) + \cos(2\pi t)) \) as even/odd components,
  \[ x_e(t) = \frac{1}{2} \left[ x(t) + x(-t) \right] = e^{-5t^2} \cos(2\pi t) \]
  \[ x_o(t) = \frac{1}{2} \left[ x(t) - x(-t) \right] = e^{-5t^2} \sin(2\pi t) \]

- The above example demonstrates some properties of even/odd signals.
  - The product of two even signals is even.
  - The product of an even and an odd signal is odd.
  - The product of two odd signals is even.
* Impulse Function

Another important signal is the Dirac Delta function, aka the impulse function.

\[ \delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases} \text{ where } \int_{-\infty}^{\infty} \delta(t) \, dt = 1 \]

The impulse models a "kick" to the system.

- Mathematically, the impulse is not a function, but a distribution. We can gain some appreciation of this by looking at some equivalent \textit{def's}.

\[ \delta(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \text{rect} \left( \frac{t}{\varepsilon} \right) = \begin{cases} \frac{1}{\varepsilon} & \text{ for } \varepsilon \text{ at } t \\ 0 & \text{ else} \end{cases} \]

\[ \delta(t) = \lim_{\varepsilon \to 0} \frac{1}{\sqrt{\pi \varepsilon}} e^{-\frac{t^2}{\varepsilon}} \text{ the Gaussian function} \]

\[ \delta(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ \frac{\sin \left( \frac{\pi t}{\varepsilon} \right)}{\frac{\pi t}{\varepsilon}} \right]^2 \text{ the sinc}^2 \text{ function} \]

- Two useful properties of the impulse function:

  - Sampling \( f(t) \delta(t) = f(0) \delta(t) \)

  - Sifting \( \int_{a}^{b} f(t) \delta(t) \, dt = f(0) \) for any \( a < 0 < b \)

Q: Would you consider \( \delta(t) \) even or odd?
# Unit Step or Heaviside Function.

We have already been using this function,

\[
U(t) = \begin{cases} 
0 & t < 0 \\
1 & t > 0
\end{cases} \quad \text{or} \quad U(t) = \begin{cases} 
1 & t = 0 \\
1 & t > 0
\end{cases}
\]

- \(U(t)\) models a switch

\[
f(t) = \begin{cases} 
0 & t < 0 \\
1 & t = 0 \\
0 & t > 0
\end{cases} \quad \text{or} \quad f(t) = U(t)
\]

- The unit step can be defined in terms of the impulse and vice-versa

\[
U(t) = \int_{-\infty}^{t} \delta(r) \, dr \quad \text{and} \quad \delta(t) = \frac{dU}{dt}
\]

- Using the notion of distributions.

**E.g.**

\[
\delta(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} e^{-\frac{t^2}{2\varepsilon^2}}
\]

**Demonstration**

\[
U(t) = \int_{-\infty}^{t} \delta(r) \, dr = \lim_{\varepsilon \to 0} \int_{-\infty}^{t} \frac{1}{\sqrt{\pi \varepsilon}} e^{-\frac{r^2}{2\varepsilon}} \, dr
\]

\[
= \lim_{\varepsilon \to 0} \frac{1}{2} \left( 1 + \text{erf} \left( \frac{t}{\sqrt{2\varepsilon}} \right) \right)
\]

- We can also use the unit step to define other signals.

**E.g.** Ramp

\[
R(t) = \int_{-\infty}^{t} U(r) \, dr = t \, U(t)
\]

**E.g.** Pulse

\[
P(t) = U(t) - U(t - \varepsilon)
\]

a pulse of length \(\varepsilon\)